Frans van Galen & Dolly van Eerde (eds.)

Mathematical investigations for primary schools

http://www.fisme.science.uu.nl/en/impome/

This booklet will become thicker

This is not the final version of the booklet. We hope to include more investigations in the coming months.

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Mathematical Investigations for primary schools

Dolly van Eerde and Frans van Galen

1. What is in this booklet?

This booklet contains a set of inquiry mathematics lessons. Inquiry lessons promote students to actively participate in investigating non-standard problems. They promote students to pose questions, to think and discuss about different solutions and to develop deep understanding. The problems that are described in this booklet are intended as starting points for what we like to call 'mathematical investigations'. The goal of the lessons is not so much that students will find a solution for the given problem, but that they investigate the underlying mathematical ideas. The inquiry lessons in this booklet can be used as a complementary to the regular mathematics lessons where standard problems, standard solutions and exercises often are predominant.

The activities are selected from master theses of students participating in the IMPoMe project (Indonesian Master Program on Mathematics Education). This project was a cooperation between two Indonesian Universities - Universitas Negeri Surabaya (Unesa) and Universitas Sriwijaya (Unsri, Palembang) - and the Freudenthal Institute of the University of Utrecht (the Netherlands). The activities are taken from lesson series that were designed by the students while they studied in Utrecht, and then tested in Indonesian schools. The complete theses can be found at http://www.fisme.science.uu.nl/en/impome/

The investigations we have chosen are intended for different grades in primary education and the first grades of secondary education and they encompass a variety of mathematical domains. Together they give an impression of what is meant by 'inquiry lessons' or 'mathematical investigations', and by 'Realistic Mathematics Education'.

We hope that this book will be useful for teacher education. The activities may be used to let prospective teachers practice a way of teaching in which the ideas and inventions of the children themselves are the most important. It may also be used by master and bachelor students as a resource book with ideas for lesson designs and research. Everybody should feel free to redesign and adapt the lessons for a specific target group of students.

2. Characteristics of inquiry based lessons

Different types of lessons

Not every mathematics lesson has to be an inquiry lesson. There are different kinds of lessons in mathematics education and they all serve a certain purpose. Lessons, for example, in which students practice skills are very important. Inquiry lessons are useful especially when students explore a new topic or explore the relations between topics, when the emphasis is on building understanding and insight.

For most children, inquiry lessons will be something they are not familiar with. Teachers could carefully start with giving an inquiry lesson once in a while and then do it more often when the students - and the teacher herself[1] - become familiar with such lessons.

Good problems

Inquiry lessons in primary education and the first years of secondary education almost always start with a context problem. The emphasis then is on the ideas that the children develop while discussing such problems. Students must therefore be given much time to think, to explore different solutions and to discuss their ideas.

What is a good problem? A good problem is a meaningful and interesting problem that stimulates students to think. A good problem is an open problem that cannot be solved by a standard procedure and therefore invites students to come up with their own solutions. Most important is that such a problem should engage students in doing and learning real mathematics, the mathematics that is fundamental in the problem. The activities we have collected in this booklet are examples of such problems.

Another way of teaching

Teachers often see their role as someone who explains - as clearly as possible - the ideas of mathematics for the students. In inquiry lessons, however, the teacher's role is to stimulate students to reinvent these ideas themselves. The teacher helps students to formulate their own ideas and helps them to understand how other children have reasoned. This implies a switch in

1. It is common usage to use the male forms 'he' and 'him' when a text applies to both male and female teachers, but we have chosen to use 'she' and 'her'.

the role of the teacher and the students.
The teachers role changes from telling and explaining, to asking questions, to listening and trying to understand how students reason. And to guiding their learning process. The role of the students changes from passively listening and trying to give 'the right answer', to being more active by asking questions and by participating in the discussion on how to solve the problem. The students should do as much thinking and talking as the teacher.
In short, in inquiry lessons, when students do mathematical investigations, the focus is on the learning of the students. The teacher presents a problem to the students, and then steps back for a while, stimulating the students to do the thinking. The teacher should give support by answering informative questions and by giving hints and scaffolds, but not by presenting the solution.

Structure of an inquiry lesson

Inquiry lessons can be organized in different ways, but the general pattern is as follows. The teacher starts with an *introduction of a problem* with the whole class. She discusses the context, asks questions that help students to understand what they have to figure out and she allows students to ask questions. Questions that already would lead to specific solutions, she postpones to later, with remarks like: 'Let us all think about the problem ourselves first".
Then students start *working in small groups* on the problem. Working in groups is important because students will hear different solutions and argumentations, and can practice to explain their own solution and argumentation. Some students may see a solution almost immediately, while others will need more time. It is good practice, therefore, to give students some time to think individually - one or two minutes in silence - before they start a discussion within their group. This allows all group members to bring in something. A clear presentation of the problem and questions on a worksheet helps the students to focus. There should be space on the worksheet for drawings, schemes, computations and for writing down ideas in words.
When the students have discussed and exchanged their initial ideas, the teacher asks a few groups to explain their approach so far. This will secure that every student really understands what is asked, and it helps all groups to make a start.
Students then *continue working in groups*, while the teacher walks around to observe what the groups do, and to answer and pose questions. Walking around allows the teacher to get an impression of the work of all groups and to select groups with different solutions.
Finally the teacher invites some groups in front of the class for a *presentation of their solution*. Each group member should be able to explain the solution the group has chosen. The other students in the class ask questions to the presenting groups and give their opinions. The teacher *guides the discussion*.

How to prepare for an inquiry based lesson?

As a teacher, the best way to prepare for an inquiry lesson is first to imagine oneself in the position of the students: if you would be a student, with a lot of practical knowledge, but not the formal knowledge that an adult may have, how would you solve the problem? When you take some time for this, you may find different ways to do it, so probably also your students will come up with different solutions. Then think about:
- what students might find difficult in interpreting the problem,
- how students may use different approaches to solve the problem,
- how the problem could promote students learning in reaching the learning goals,
- how you as the teacher may stimulate and guide the students in their learning process.

On a more practical level this means that you have to think about:
- how to present the problem in such a way that students will understand it and will want to solve it,
- what hints and scaffolds you will give when students do not know how to proceed,
- what you choose as topics for the whole class discussion.

3. What is RME/PMRI?

Realistic Mathematics Education (RME) and Pendidikan Matematika Realistik Indonesia (PMRI) can be characterised in different ways. It can defined by characteristics like:
- the use of meaningful contexts,
- the building on students' informal knowledge,
- the emphasis on interaction,
- the use of models.

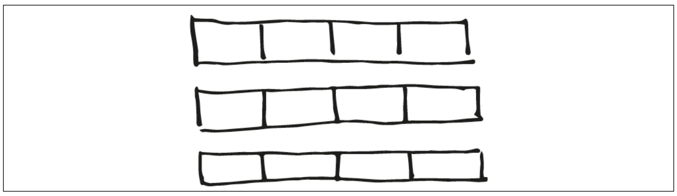
RME/PMRI can also be characterized as promoting mathematics education based on *guided reinvention* (Freudenthal 1999). This principle in a sense covers all the forementioned characteristics: learning starts with problems within a meaningful context, children come up with their own solutions for these problems

and this then leads to a discussion on the mathematics behind the problems. Schematic representations that are used to solve concrete problems may be generalized and used as tools to solve new problems. In this process the teacher plays a central role, not by explaining the mathematics, but by asking questions and guiding the students to discussing fundamental mathematical issues.
In stead of RME/PMRI we may also speak of inquiry based mathematics. In order to illustrate the aforementioned characteristics, we very shortly discuss three examples of mathematical investigations.

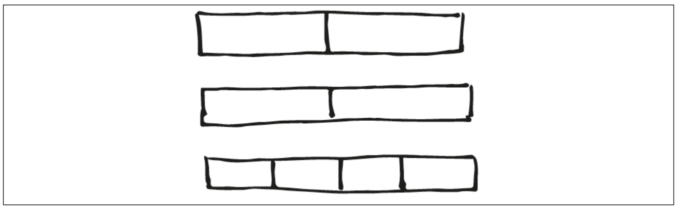
Example 1: Fair sharing

The context of fair sharing can be used to let students explore the relations between fractions. All children are familiar with situations in daily life where they have to share food or something else, and they know that this should be done in a 'fair' way. So if we ask the students how four children may share three sandwiches (or a local delicacy) they may come up with solutions like the following (van Galen & van Eerde, 2013).

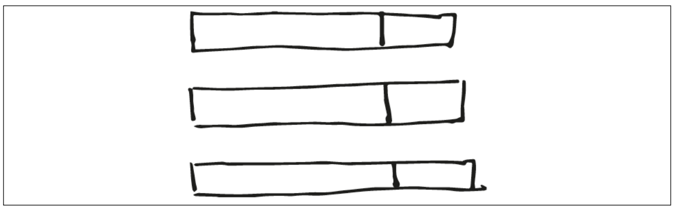
You may cut each andwich in four parts:



You may give each child half of a sandwich and then divide the remaining sandwich in four:



Three children cut off one fourth of their sandwich and give this to the last child:



It is important, however, that students not only find ways of solving the sharing problem, but also describe their solutions in words. For example: 'We divided every sandwich in four parts. Each child gets one fourth from each sandwich, so they get three fourth.' And also with fractions. For example:

- 3/4 sandwich = 1/4 sandwich + 1/4 sandwich + 1/4 sandwich
- 3/4 s = 1/2 s + 1/4 s
- 3/4 s = 1 s - 1/4 s

This example illustrates how students may use their own, informal knowledge about fair sharing to solve a context problem. The exploration of relations between fractions then becomes a starting point for building a network of relations between all kind of fractions.
Letting students explore problems of sharing sandwiches, pizza's or other objects is quite different from introducing a ready-made fraction model - like the fraction circles - and to use this model to explain relations between fractions. In that case teaching starts at a quite formal level, which means that for students the focus will be on calculation rules and on procedures, not on insight.

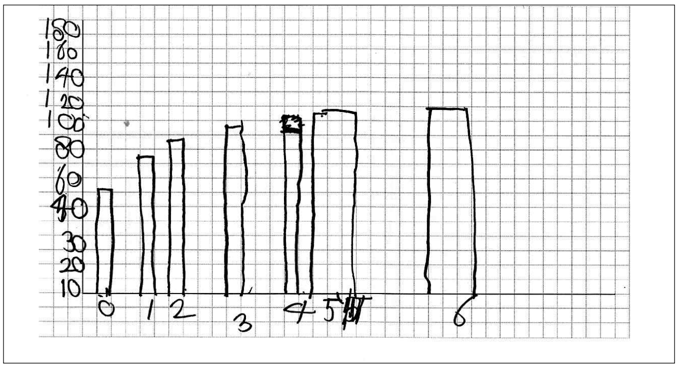
Example 2: What makes a graph a useful graph?

0 jaar	52 cm
1 jaar	74 cm
2 jaar	88 cm
3 jaar	97 cm
4 jaar	105 cm
5 jaar	112 cm
6 jaar	118 cm

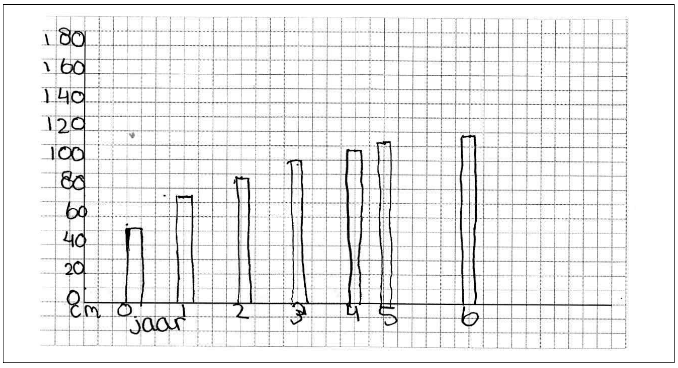
(from: van Galen and Markusse, 2018)

The teacher tells how the father of Mara has measured Mara's length each birthday. The students are asked to make a sketch or a graph that shows how Mara has grown in these years. If this problem is presented in an open way, some students may make drawings of actual children, ordered in length from small to big. If students, however, have encoutered graphs before, they may come up with drawings like those on the next page.

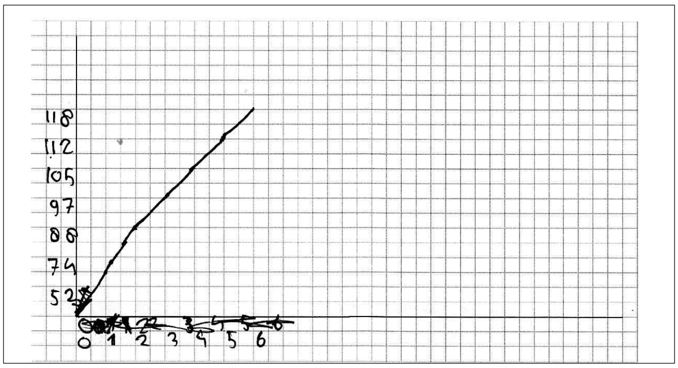
One could say that all graphs are 'correct' in the sense that the given values for age and length are represented in the graphs. Ger, Thijs and Sabine, however, have yet to reinvent a fundamental idea: the scales for length and time should represent the ratios within the measurements. Ger starts with steps of 10 on the



Ger



Sabine



Thijs

Natja

vertical axis and then changes to steps of 20. Sabine is correctly on that point, but she and Ger do not respect the ratios on the time axis. Thijs simply puts all given values next to each other on the axes, which leads - not very useful - to a straight line. Natja, finally, respects the ratios on both axes, but obviously believes that graphs should start in the origin. (There is of course, a value of 0,0, but that lies 9 month before birth, not one year).

A problem like this should be given to students who already have some knowledge about graphs. They may have interpreted given graphs before, but do they really understand the ideas behind graphing? The best way to test this is to let students draw their own graph for a given context situation. Exploring the differences in the student work will lead to a fruitful discussion about what makes a graph a useful graph.

Example 3: The context of mathematics

Our third example is meant to illustrate that a meaningful context does not necessarily have to be a context from daily life. The context may also lie within mathematics itself, as in the problem at the right. Although the problem is about bare sums it will have meaning for children, especially if they discover how each subtraction can be solved quite easily if you use the fact that the numbers are just above and just below a nice round rumber. The problem can be solved in different ways, by taking away the second number from the first or by 'counting up' from the second to the first number. These approaches can also vary in the way students apply them. Students could use a mental approach and successively subtract units, tens and hundredths or do a traditional long subtraction on paper.

Students who apply the traditional way to subtract might switch to the shorter and easier way when they become aware of the small differences between the two numbers.

Solve the following problems and write down how you did it.

72-69 Answer: ....
Solution: ....
1002-998 Answer: ....
Solution: ....
2003-1996 Answer: ....
Solution: ....
9002-8099 Answer: ....
Solution: ....
Can you make more of this kind of sums?
My sums:
....
....
....

Interaction

Interaction is crucial in inquiry based mathematics. Quite often the interaction in a classroom is limited to the teacher asking questions and students giving answers. This traditional interaction pattern is know as the IRE pattern: the teacher *initiates* the interaction with a question, the student gives an answer - a *response* - and the teacher *evaluates* by saying whether the answer is good or not. This kind of *vertical interaction* is limited to interaction between the teacher and one student.

In inquiry based mathematics, however, there is not only interaction between teacher and students, but also between the students. We call this *horizontal interaction*. The teacher may function as an intermediary within this discussion, rephrasing ideas of a student and asking for reactions. The interaction between students and teacher will also be different in inquiry based mathematics: the teacher gives students the opportunity to formulate their ideas and reacts to what they bring in.

The most important aspect of the interaction between teacher and students is perhaps that the teacher should really want to understand how students think. Here, like in every conversation, what one hears may be influenced strongly by one's expectations. To avoid filling in what the student is thinking - and missing sometimes what the student really wanted to say - follow-up questions are useful. When the students become accustomed to such questions, they will become better and better at answering them.

The interaction should start right from the beginning, when the teacher introduces a context problem. Preferably she tells the story in her own words, and she checks if the students understand what the problem is. Then she asks the students to think and discuss about ways to find a solution, preferably in small groups. Working in small groups has many advantages. Students often feel more secure in a small group and this will promote their participation. They can share their thinking, listen to others and cooperate in finding a solution. Group work greatly enhances students' opportunities for learning, as learning is a social process: students learn from the teacher and from each other.

When students have had some time for discussion in their small group the teacher may ask one or two groups to tell about their approach so far. After that the students continue the discussion in their groups. Finally some groups present their solutions followed by a whole class discussion.

Whole class discussions are the heart of mathematics education. Of course, practice is important too, but learning in the sense of adapting one's ideas and making new discoveries, takes place primarily during class discussions. If it is going well, such a discussion takes place largely between the students themselves. The teacher plays a key role, however. She mediates, for example, by reformulating the children's suggestions, and she ensures that the discussion concerns the points that are important, but preferably using the ideas of the students as a point of departure.

The teacher must always show that it is important that everybody really understands what a student is contributing. She will remind the students regularly that not only the teacher, but everyone in the class must be able to follow the reasoning of a student. Such a classroom culture is necessary if we want children to become interested in the reasoning of other students, and try to compare this with their own ideas. Children are extremely perceptive about whether their teacher is truly interested in what they are thinking, or instead, she is simply waiting for a specific solution she has in mind.

4. How to use the lessons

The mathematical activities are meant both for primary and secondary mathematics education. They can be used in different phases of teacher training and by teachers during mathematics lessons.

Teachers and future teachers can select an investigation and do this with a whole class of students. However, it could be sensible to first do a tryout with a small group of students. This enables the (student) teacher to experience students' difficulties, questions and solutions they come up with. In teacher training settings the teacher trainer could start with giving the student teachers the lesson and ask them to look at the problems through the eyes of the children. They could work in groups imagine what difficulties children could have, think up the solutions the children would come up with and discuss these in a whole class discussion. This is a rich way for teachers to prepare a lesson.

5. Overview of the mathematics lessons

The activities and lesson series are structured around the following mathematical topics:

- Measurement& Geometry (unit of measurement, area of different shapes, spatial structuring)

- Numbers (addition and subtraction, fractions, percentages, decimals, negative numbers)
- Proportions
- Relations (coordinate system, graphs, algebra)

Sources

van Galen, F.H.J. & van Eerde, H.A.A. (2013). Solving Problems with The Percentage Bar. *IndoMS Journal on Mathematics Education*, 4 (1), (pp. 1-8) (8 p.). http://jims-b.org/?m=201212

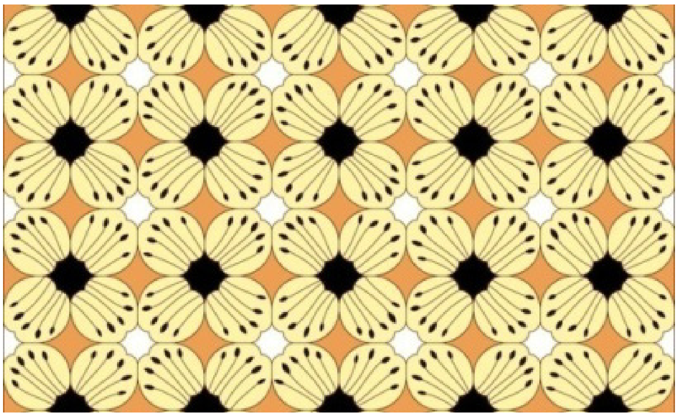
F. van Galen & A. Markusse (2018). Rekenen met verhoudingen op de basisschool; Reken-wiskunde-didactiek voor de bovenbouw. Groningen/Utrecht: Noordhoff Uitgevers

Developing the notion of symmetry with Batik patterns

Cici Tri Wanita

The activity

The teacher tells about a gallery that is organising a Batik exhibition. The gallery has two rooms and the organisers want to put batiks that are similar in the same room. The worksheet shows pictures of small parts of the batiks. The students are asked to sort the patterns of these batiks in two groups and to explain the difference between the two groups. Some patterns have 'regular' patterns (line symmetry) others are 'irregular' patterns (no line symmetry).
Then the students are given small mirrors they can use to explore line symmetry. Finally they are asked to find the axes of line symmetry and the diagonals in the patterns.



Materials

The worksheet, small mirrors, ruler, pencil.

For which students is this activity?

According to the Indonesian 2006 curriculum, symmetry should be a topic in the second semester of grade 5.

Why is this an interesting activity?

The activity uses traditional Indonesian Batik art to let students explore the concept of symmetry and the different types of symmetry . The patterns are familiar to the students, which makes Batik art a good context for introducing geometric concepts. At the same time the activity promotes children to appreciate the value of this typical Indonesian art form.

What to expect and how to support the students?

Sorting the pictures while working in groups will probably not be difficult for the students who use their sense of symmetry. The pictures A, D, G, H, I, J and L have a regular pattern, whereas the other Batiks are irregular. However, some students may connect regularity to repeating motifs in the pattern and incorrectly include some patterns with repeating motifs but no line symmetry in the regular group.
Formulating what the regular group has in common may be a challenge for the students. There is some repetition in A, for example, but left and right side of this piece of Batik are not exactly the same.
During the whole group discussion the students exchange their ideas. Probably some students will refer to the concept of symmetry and maybe the effect of a mirror. The teacher explains what an axis of symmetry is and let students explore how a small mirror can help to find such an axis of symmetry in the patterns. Some children may hold the mirror at the edge of the picture initially. The teacher can show that this doubles all pictures and aks if there will be a difference between the two groups of patterns. The teacher clarifies that the picture should remain exactly as it is without the mirror.
Once the students find one axis of symmetry they will discover that the patterns have more than one axis of symmetry. In all the regular patterns the mirror can be placed in the middle, both vertially and horizontally, and in most of the pictures the mirror can also be put diagonally. However, students should become aware that the diagonals are not always axes of symmetry.

A longer series of lessons

In the follow-up activities of the study, the students learn about rotational symmetry and not only study Batik patterns with line symmetry, but also with rotational symmetry. Later they learn how to change an asymmetric batik pattern into a symmetric one and how to complete an unfinished symmetric pattern based on the given axes of symmetry. Finally the students learn how to create a batik design with rotational symmetry.

Sources

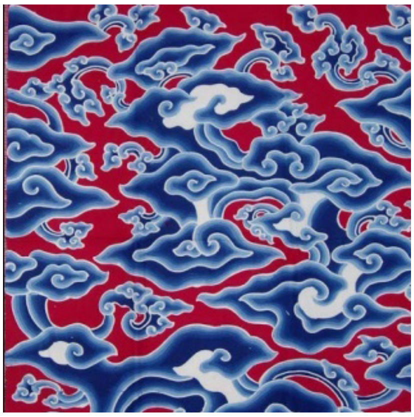
Cici Tri Wanita. *Developing the notion of symmetry through batik exploration*. Master thesis Universitas Negeri Surabaya, 2014.
www. fisme.science.uu.nl/en/impome/

A

B

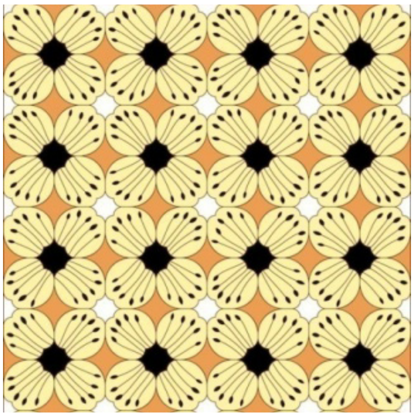
C

D



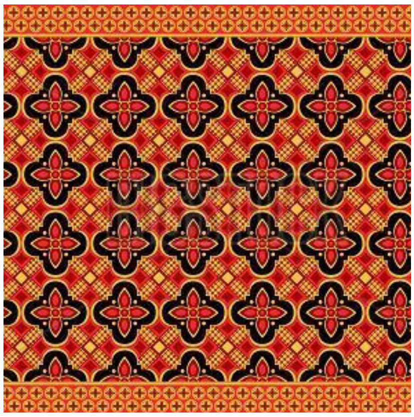
E

F



G

H



I

J

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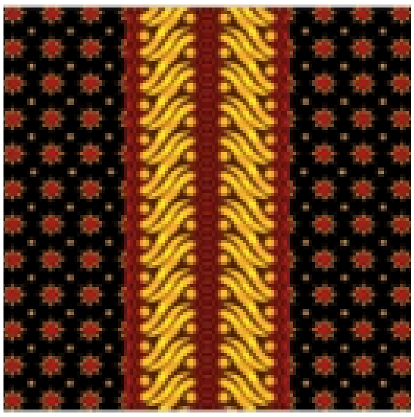


L

Spatial visualization and volume measurement

Shintia Revina

The activity

The children work in small groups. Each group is given two boxes and the students have to decide which of these is 'bigger' in the sense of having more capacity. After some initial discussion the children are given some smaller objects of a certain type and they try to figure out how many of these will fit into the bigger boxes. In the study students used pieces of 'dodol' (traditional Indonesian sweets) to compare the volume of two boxes, but the task can also be done with, for example, empty tea boxes of the same shape and size, match stick boxes, or lumps of sugar. The boxes students have to compare should have a corresponding size. The students are given just a few objects - measurement units - not even enough to fill up the bottom of the boxes; they have to reason in terms of imaginary layers of objects.

Materials

- For each group two boxes. Length, width and height of the boxes are different, but where one box is wider, the other box should be higher.
- Each group is also given a few pieces of 'dodol' or other cuboid-shaped objects, enough to check how many fit in the length and width of a box, but not enough to fill the bottom completely. It is not necessary that the objects will fit neatly into the boxes; some space may be left open.

For which students is this activity?

According to the 2006 curriculum, volume is a topic for the 1st semester of grade 5. If, however, students in higher grades still struggle with the concept of volume, it may be useful in these grades too.

Why is this an interesting activity?

In order to find an answer the students have to visualize how each box can be filled with layers of smaller objects. They then have to calculate the number of objects in each layer and from this the total number of objects for each box. An activity like this offers a concrete basis for understanding what is meant by 'volume'. Instead of telling students that volume is 'length x width x height' we should give children the chance to reinvent this calculation procedure. Of course students will not be able to do this on their own; the teacher should guide and support them. The activity that is described here could be the first step in such a process.

What to expect and how to support the students?

The problem is about volume or capacity. Students may think that this asks for calculations with centimeters and so they may want to use a ruler. Tell them, however, that they will have to solve this problem without a ruler.

If the students are given concrete objects - pieces of dodol, little boxes, sugar lumps - the idea of filling up the bigger boxes may come up quite naturally. The fact that each group will get just a small number of objects may create a problem, however. Allow the students some time say, 5 minutes to discuss the problem and to find a solution. Then discuss with the whole group what was found in the groups and what general approaches are possible. All useful approaches will in some way analyse the situation in terms of a number of layers. If everyone agrees, let the groups continue their work.

The students will probably use different ways to calculate the number of objects in one layer:

- Counting: moving one of the objects over the bottom of the box and counting how many will fit in total;
- Repeated addition: figuring out how many objects will fit in one row, and then add this nummber for each next row;
- Multiplication: if each row consists of 6 objects and 4 rows will fit on the bottom, then the total number will be 6x4.

Similar approaches may be used for calculating the total number of objects that will fit in the box: repeated addition and multiplcation.

After the groups have found a solution, discuss with the whole class what approaches or strategies have been used and discuss their effectiveness.
Boxes are easy to fill with cubes and blocks that have an equal length, width and height. Other objects may cause problems, as they sometimes have to be put in different directions to fill up the bottom of the box in the most efficient way. (This, in fact, is the reason why we use cm^3 or other cubic units for measuring volume.) Encourage the students to describe even such arrangements in terms of multiplications, like: '2 rows of 6 and 2 rows of 2 in the other direction'.

A longer series of lessons

This activity was designed as the introductory activity for a series of 7 lessons. Students need at least a few lessons like this before the formula for volume measurement - volume = length x width x height - should be introduced. In fact the lessons are designed to stimulate children to reinvent this formula themselves. This will result in a deeper understanding, preventing that students just memorise the formula and apply it without any insight.
In the follow-up activities of the lesson series questions are asked about pictures with 3D-arrangements of objects and the students are asked to make drawings of such 3D-arrangements. In later problems wooden blocks are used instead of objects from daily life. Different sizes of blocks are used, which helps the students to realize that the number that describes volume or capacity is related to the size of the unit that is used.

Sources

Shintia Revina. *Design research on mathematics education: spatial visualization supporting students' spatial structuring in learning volume measurement.* Universitas Sriwijaya, 2011

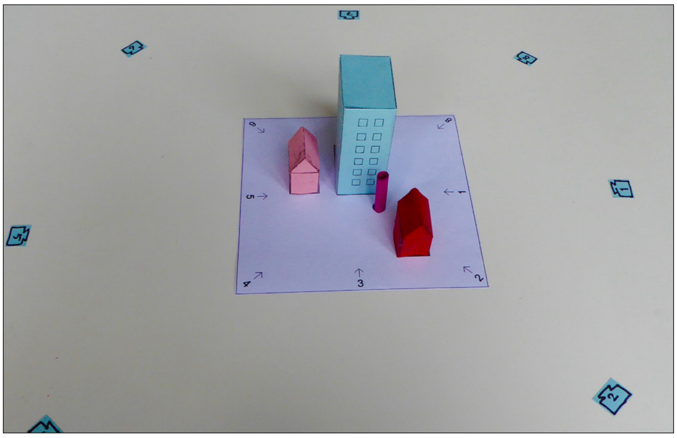
http://www.fisme.science.uu.nl/en/impome/

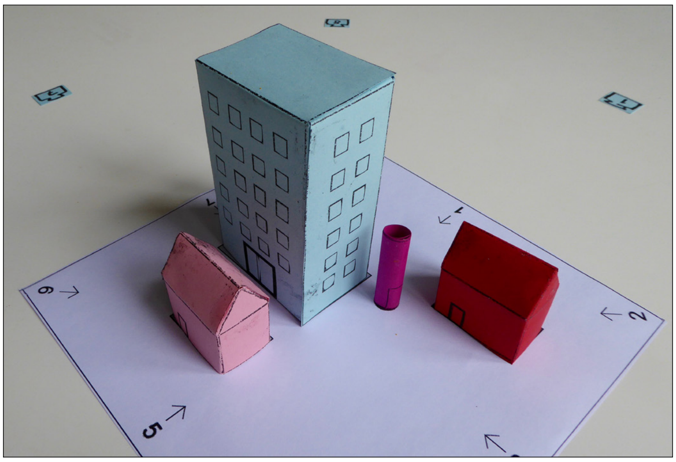
Understanding representations of three-dimensional situations

Aan Hendroanto

The activity

Four miniature buildings - an apartment building, two small houses and a round tower - are placed on a piece of paper that specifies their positions. On the paper eight arrows indicate eight different directions to look at those buildings.
The children are asked to help a photographer to make pictures of these buildings. He wants to make a picture on which all of the buildings are visible and so the question becomes from which direction this is possible. As a second task the students are given 8 pictures made from different spots and asked to decide which picture was made from which direction.
The children work in small groups. They are given paper 'model cameras' - the outside halves of a match stick box - as tools and they may check their answers with a real digital camera.





Materials

- Paper models for the buildings, see the other page. You can make a set for each of the small groups, or you can make just one or two sets and let the groups work at separate times.
- Worksheet 1. Cut off the ground plan and give the students the upper part.
- The task will be more clear if you glue little cards with the picture of a numbered camera at some distance around the ground plan (see pictures)
- Worksheet 2.
- The outside halves of matchstick boxes or empty toilet paper rolls. Children can look through these and use them as a little 'model camera'.
- Optional: a real camera to check the answers.

For which students is this activity?

The activity is an adapted version of the first activity in a lesson series that was developed for children in grade 3. It is also suitable for children in higher grades.

Why is this an interesting activity?

In the Indonesian curriculum so far not much attention is being paid to spatial ability. The focus in geometry is more on 'pure mathematics', for example the characteristics of 2D-shapes and 3D-solids. Reasoning about the world around us in a mathematical way deserves much more attention.
A picture is a 2D-representation of a 3D-situation. By letting children explore the relation between 2D and 3D we stimulate them to develop their spatial ability. The 'model camera' in this activity helps the children to discriminate between what they know and what they see. They know in this case that there are four buildings and if we would ask them to draw the situation they will probably draw all four. Seen from a certain position, however, some buildings may be hidden behind other ones.

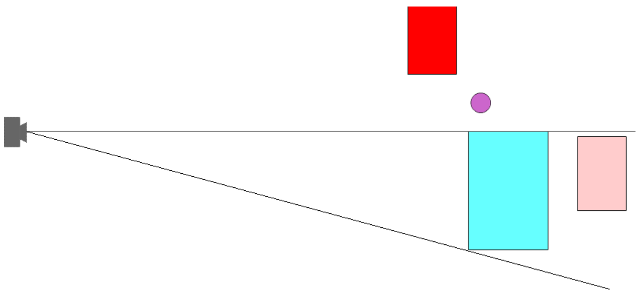
What to expect and how to support the students?

Make sure that all children understand that the photographer wants to make a picture on which all four buildings are visible and discuss why that may not be the case from every postion. Also discuss that the photographer will make his pictures at street level; seen from the air the four buildings will be visible from every direction.
Although some of the students may think otherwise, most of them will need to go and look from the specified directions, using a model camera. The

model camera's supply a viewing frame and they force the children to use only one eye; this makes the view more picture-like.
When the students have done worksheet 1, pay attention to the language for describing the position of the houses. The students should use terms like 'left', 'right', 'in front of' and 'behind'; for example: 'the round tower is at the right of the apartment building'.

After they have finished worksheet 1 the students are given worksheet 2. Sorting the pictures with the different directions should not be difficult after the foregoing discussions.

Although it is not the purpose of this activity to discuss explicitely the idea of vision lines, be alert if children use this notion. 'Vision lines' are lines that can be drawn from the eye to a certain object. In a top view, for example, the lines from the eye to the edges of an object show what will be hidden behind that object.



These vision lines show how the little pink house is hidden behind the apartment building.

A longer series of lessons

In the teaching experiment this was the first, explorative activity within a longer series. In the activities that followed 3D-buildings of wooden blocks were used. In some tasks the students had to compare pictures and to say from which stand point they had been made, in other tasks they had to reconstruct a building of blocks on the basis of a set of pictures of that building.

Sources

Aan Hendroanto. *Developing students' spatial ability in understanding three-dimensional representations.* Master thesis Universitas Negeri Surabaya, 2015.
www. fisme.science.uu.nl/en/impome/

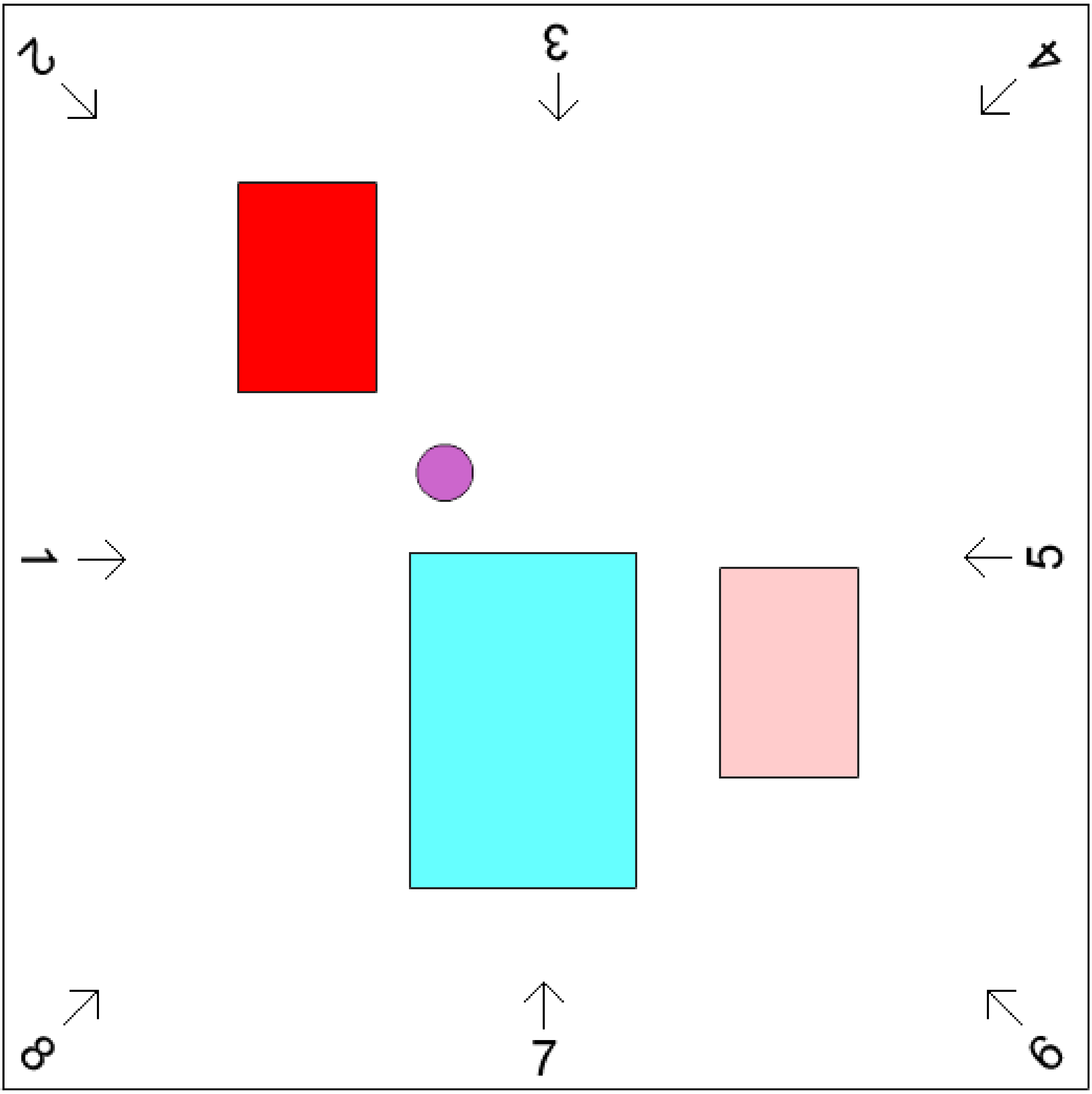
Worksheet 1

Toni, a photographer, wants to make a picture of the buildings. It should be a picture on which all four buildings are visible. He wonders what whould be a good spot for making such a picture.

1. Walk around the paper houses and use a ‘model camera’. Write down for each spot (1-8) which buildings can be seen from that spot. Or, in other words: which buildings will not be hidden behind other buildings.

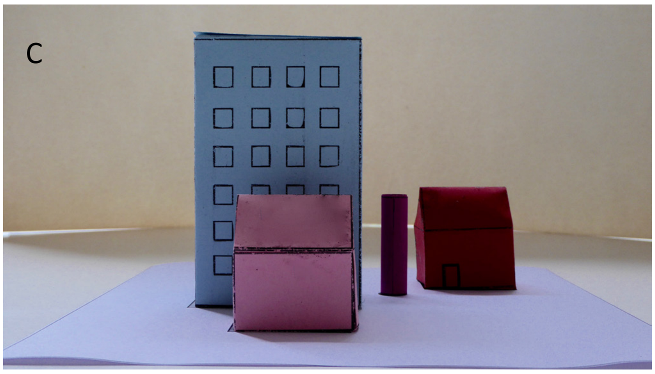
2. What would you tell Toni, the photographer?

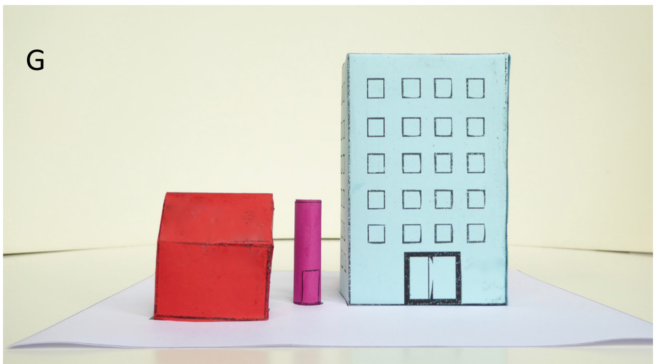
3. Choose one of the spots and write down as precisely as possible how the buildings will be shown on a picture. Use terms like ‘right of’, ‘left of’, ‘in front of’ and ‘behind’.

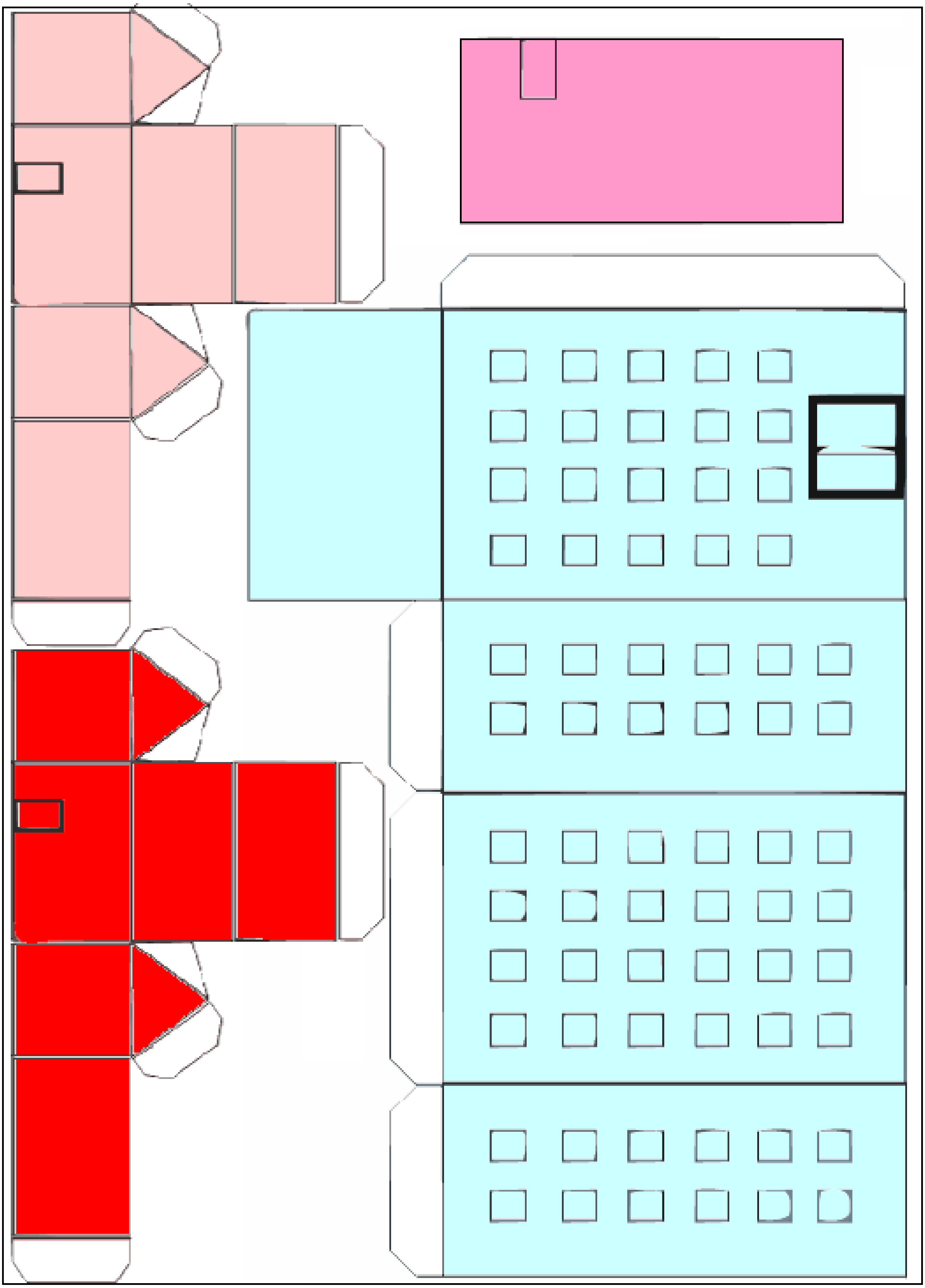


Worksheet 2

The photographer made 8 pictures.
Write down which picture (A - H) has been made from which direction (1 - 8).





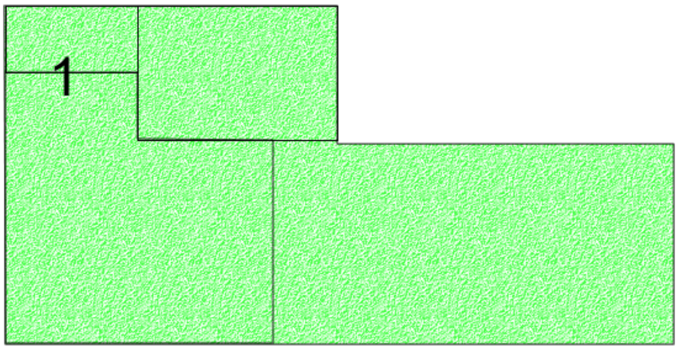


Developing understanding of area through reallotment activities

Wahid Yunianto

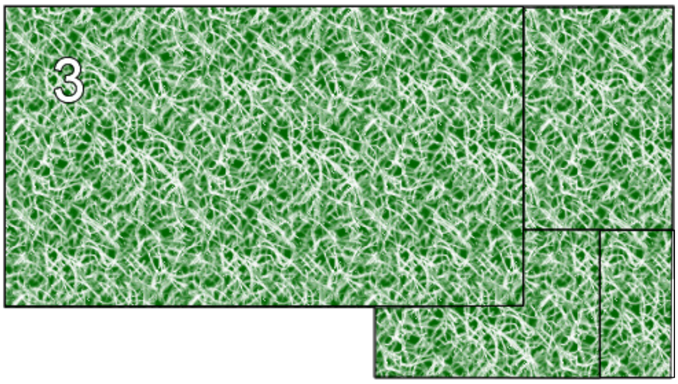
The activity

The teacher tells about a farmer who owns a rice field situated between two factory buildings. The factory wants to use his land and offers the farmer one of six other rice fields in exchange. Which of these six rice fields should the farmer accept? To answer this question, the students cut out the rice fields 1 to 6 on the worksheet and compare these with the old rice field. After the students have made a choice, they are asked to compare the length of the paths around the new field and the old field.



Materials for each group of students

- The two worksheets and scissors for question 1. They may be printed in black and white. Optional: glue.
- A piece of string for question 2. Optional: needles and Styrofoam.

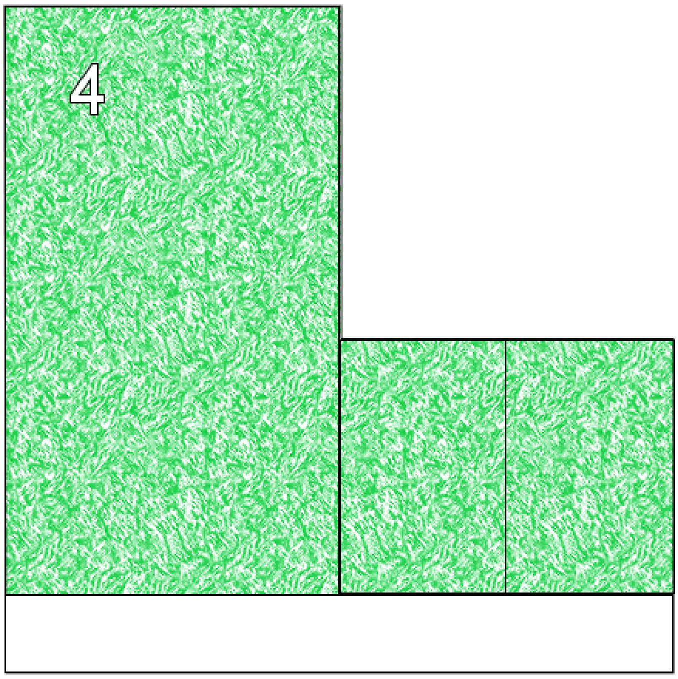


For which students is this activity?

This relatively simple problem is suitable for students in grade 4 and older. The study, which involved also more difficult problems, was done with students in grade 7.

Why is this an interesting activity?

Before students learn how to compare areas with a measurement unit like cm², they should explore how areas can be compared in a more direct way. In this activity that can be done by cutting up an area - literally or as a matter of speaking - and rearrange the parts in a different way. This is called ‘reallotment’. The second question helps to clarify the difference between area and perimeter. Often students think that a larger area must also have a longer perimeter, but - within margins - area and perimeter can vary independently.



What to expect and how to support the students?

If students cut up the pictures of the rice fields and lay out the parts on the old field, they should conclude that field 1, 3 and 6 have exactly the same areas as the old field. Field 5 is smaller, but field 2 and 4 are even larger than the old field. So there is not just one correct answer: except for field 5, all fields are a fair exchange. The pictures show some of the solutions. The core of this activity is the comparing of areas; the question about the path around the fields is just an extra question that helps to clarify the difference between area and perimeter. Measurement of the perimeter is easier if students are given a piece of Styrofoam plus some needles to put at the corner points. The perimeter of fields 1, 3 and 6 is shorter than the perimeter of the old rice field. The students should conclude from this that same area does not necessarily imply same perimeter.

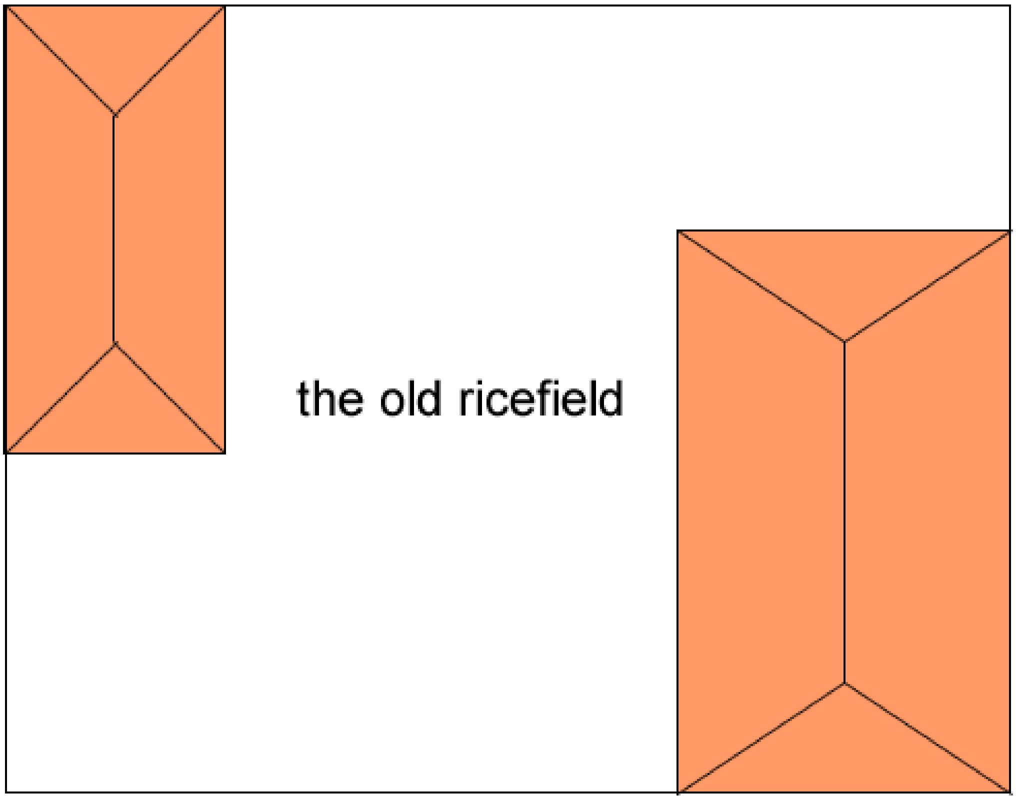
A longer series of lessons

This activity was used as the second problem in a series of 5 meetings in grade 7 that worked towards developing procedures for finding the area of parallelograms, triangles, trapezoids, rhombuses and kites.

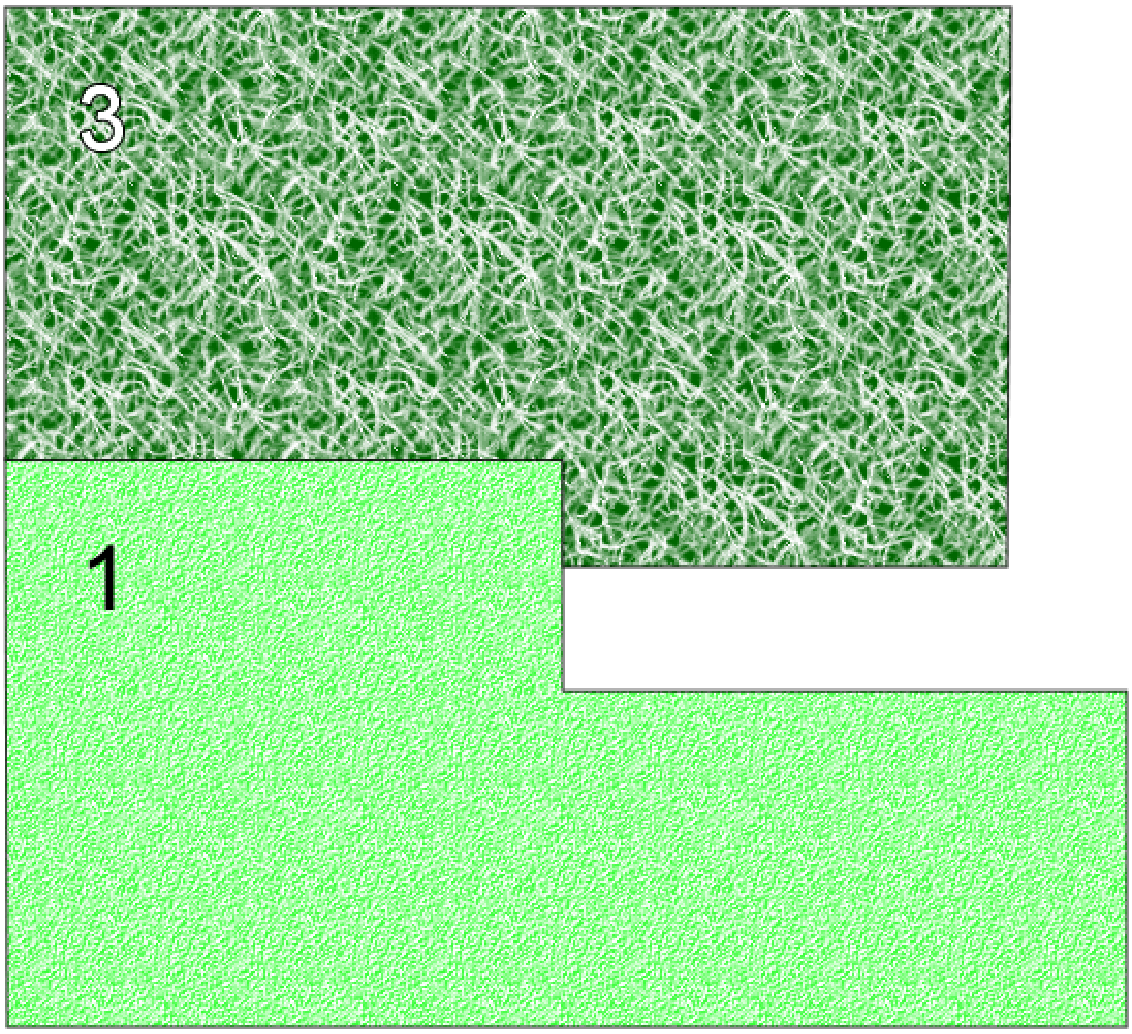
Sources

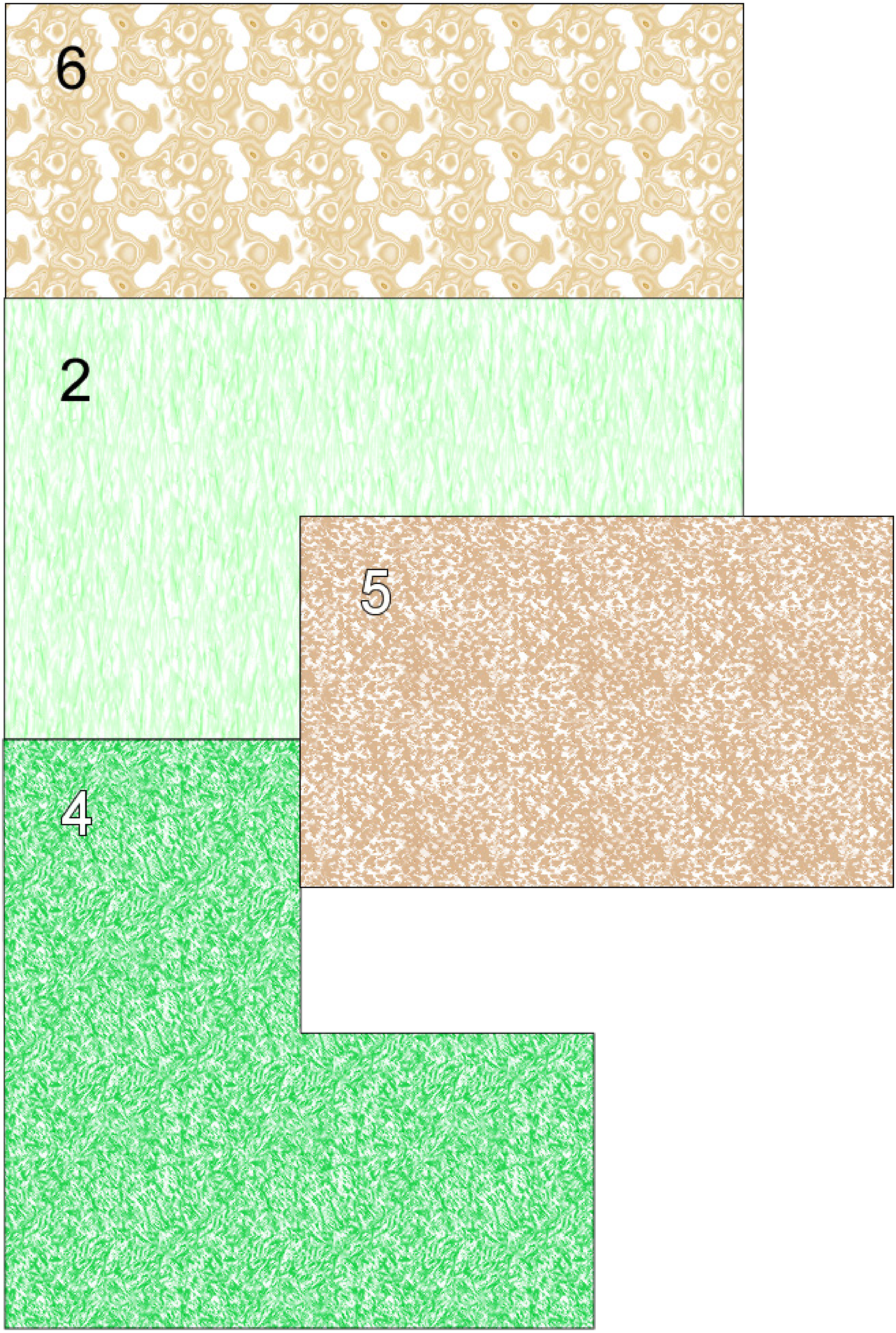
Wahid Yunianto. *Supporting 7th grade students' understanding of the area measurement of quadrilaterals and triangles through reallotment activities.* Master thesis Universitas Sriwijaya, 2014.

www. fisme.science.uu.nl/en/impome/



1. Cut out the six rice fields (1-6) and compare them with the old rice field. Which one should the farmer choose in exchange for his old rice field?
2. Now compare the field you have chosen with the old rice field. If you walk around these fields, do they have the same length? What do you conclude from this?




6
2
5
4

Introducing negative numbers

Weni Dwi Pratiwi

The activity

The students play, in groups of 5 or 6, a board game about spending money. The game is constructed in such a way that after some time all players have to borrow money from the 'shopkeeper'. Both the shopkeeper and the players have to keep track of what is earned and spend, and the real question in this activity is in fact how one should do that. The game prepares the ground for discussing negative numbers.

Materials

Game board (enlarged), play money and cards, 2 dice, 4 pieces to represent the 'shoppers'.

Rules of the 'Hop Shop' Game

- 6 persons can play: 4 shoppers, a 'bank' and a 'shopkeeper' (one shopper may also be the 'bank').
- The shopkeeper keeps record of all transactions on a piece of paper. After each transaction the shopkeeper tells how much money or debt the player has.
- The players also keep a record of how much money/debt they have.
- At the start of the game each player gets $15,-
- Each time the piece of a player passes 'Start', the bank pays that player $3.
- If, after throwing the dice, the piece of a player ends on a certain field, the player has to buy the indicated goods.
- If the player does not have the money to pay for the goods, the player can borrow money from the shopkeeper.
- If one of the players has a debt of more than $ 20,-, the game ends. The player with the smallest debt is winner.

For which students is this activity?

The lesson series was tested with students of grade 3.

The game may also be interesting, however, for older students, helping them to deepen their understanding of negative numbers.

Why is this an interesting activity?

'Debt' is a good context for a first exploration of negative numbers. If one has already a debt of $4, earning $3 can be written as -4+3, and spending $5 can be written as -4-5. In this board game it is still left to the students how they notate their gains and losses, but it is likely that students will try to use plus and minus signs in their bookkeeping. This, then, can be used for a discussion about the meaning of '+' and '-' and about the meaning of negative numbers. Students like to play board games and they will probably remember this game vividly. The context of making debts can then function as a prototype context for reasoning about negative numbers.

What to expect and how to support the students?

The teacher will start by explaining the rules of the game. The students may be given these rules on a piece of paper or the rules can be written on the board. The teacher tells the students that they have to figure out themselves how they do the bookkeeping. It is likely that all students will use the minus sign for the operation of spending money and the plus sign for gaining money. Not all students, however, will know how to represent the outcome of their calculations. Some may write 'debt $3' or something similar, whereas others will use '-3' from the start.

The class discussion after playing the game, should focus on questions like:

- What does it mean to have -3 dollars?
- Who is better off, someone with -13 or someone with -14 dollars?
- If we write a debt of 3 dollars as '-3' or '-$3',

should we not write a credit of 3 dollars as '+3' or '+$3'? (Yes we may, but it not compulsory)
- What will be the outcome of -13 -5, and what of -13 + 5?
- What will be the outcome of -3+5 and what of -3-5? And -9+12, -9-12, +12-9 and +12+9?
- Is there a difference between, for example, +$4 -$3 and -$3 + $4 ? (Yes, in this context the first number will probably be read as as the amount one has, and the second as the operation of subtraction or addition. The result will be the same, however. In mathematical terms +4-3 and -3+4 are taken as equivalent.)

A longer series of lessons

The 'hop shop game' was used as the introductory activity for a series of 6 lessons. In these lessons they explored how positive and negative numbers, and operations on these numbers, can be represented on the number line. Subtracting a negative number, like +3 -(-5), was not discussed, as this does not fit easily within the context of gains and losses. Actually it is very hard to find a real life context where this mathematical operation has a clear meaning; the meaning of +3 -(-5) is defined within the mathematical system.

Sources

Wendi Dwi Pratiwi, *Supporting students' conceptual understanding of addition involving negative numbers.* Master thesis Universitas Negeri Surabaya, 2013.
www. fisme.science.uu.nl/en/impome/

Permainan "THE HOP SHOP"

$ 1 Hop shop	$ 1 Hop shop	$ 1 Hop shop
$ 1 Hop shop	$ 1 Hop shop	$ 1 Hop shop
$ 1 Hop shop	$ 1 Hop shop	$ 1 Hop shop
$ 1 Hop shop	$ 1 Hop shop	$ 1 Hop shop
$ 1 Hop shop	$ 1 Hop shop	$ 1 Hop shop
$ 1 Hop shop	$ 1 Hop shop	$ 1 Hop shop

HOT SHOT

Go to START

HOT SHOT

Pay school books $ 6

HOT SHOT

Pay icome tax $ 5

HOT SHOT

Shopping voucher $ 5

HOT SHOT

Shop free parking

HOT SHOT

Give any amount of money to the bank for charity

HOT SHOT

Move back 6 spaces

HOT SHOT

Collect $ 1 from each player

HOT SHOT

Move forward 5 spaces

HOT SHOT

Double turn, throw the dice another time

Promoting understanding the relations between fractions

Herani Tri Lestiana

The activity

A group of students is asked to organize games for the celebration of Independence Day. They propose a competition in which children have to run with water on a plate (*lomba memindahkan air*). The team that brings most water to the bucket within a given time will be the winner. It is clear for the organizing students how they will decide who is the winner of a certain run: they will use identical, and a bit transparent buckets and put one bucket beside the other to see which one has more water. There are, however, so many children that all teams can run only once. So their question is: how can we record how much water has been collected?

The teacher steers the discussion towards making a measuring strip and using that to put marks on the two buckets. So now the question has become: how do we make a measuring strip? The teacher gives the students paper strips that are as long as the buckets's height. They are not allowed to use a ruler.

Materials

- A plastic bucket, to introduce the context. In the design study several identical plastic tubes were available which could be used for actual measurement tasks (see picture).
- Paper strips of all the same length, corresponding with the height of the bucket; probably about 25 cm. A couple of strips for each small group of students.

For which students is this activity?

The lesson can be done in grade 3, when fractions are introduced, but also later, as a way to refresh the notion of equivalence.

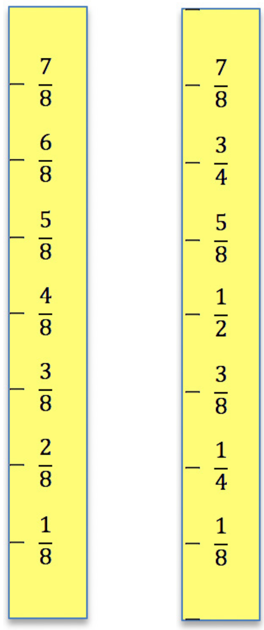
Why is this an interesting activity?

It is an open problem that can lead to a rich discussion about the relations between different kinds of fractions. The lesson may help the teacher to evaluate the informal knowledge that students already have.

What to expect and how to support the students?

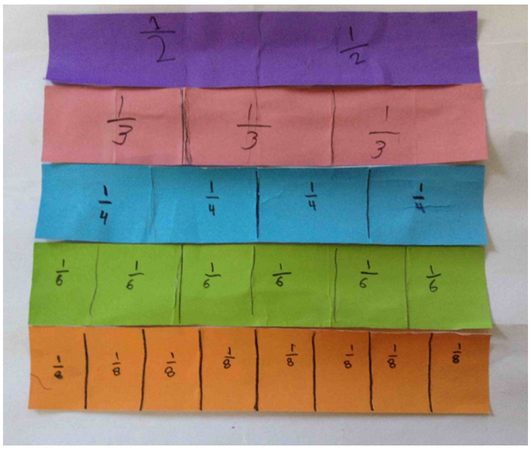
Many students probably will start to fold the strips as a way to measure off equal parts. Repeated folding will lead to 2, 4, 8 or even 16 equal pieces. Once the students have put marks on the strips, ask them to write beside the marks what each mark stands for.

If a strip is divided into four equal pieces it is possible to write 1/4, 2/4 and 3/4 beside the marks. The fraction 2/4, however, is apparently a different way of writing 1/2. In the same way the relations between eighths, quarters and halves can be discussed. This may lead to the conclusion that a strip which is divided into eight equal pieces can have different fractions beside the marks, as shown in the illustration.



Challenge students to find also ways to divide a strip, or a half of a strip, into three equal pieces. This is not easily done by folding, so some trial and error will be inevitable. The activity may lead, however, to a discussion about the relations between thirds, sixths and ninths, and between sixths and halves.

Once the students have made the paper fraction strips, they can be used for problems like: what frac-



tion is as much as 1/2?, or as much as 3/4? What is more, 3/4 or 2/3? And so on. Make sure, however, that the emphasis is on reasoning and not on a visual comparison of the strips. Not: 3/4 is more than 2/3, because the strip of 3/4 is longer than the strip of 2/3, but, for example: ‘3/4 is more than 2/3, because 3/4 is 1 - 1/4, and 1/4 is smaller than 1/3’.

A longer series of lessons

The lesson is part of a series of lessons that focus on the addition of fractions. Understanding addition and subtraction of fractions should be based on understanding equivalence. This activity offers students a way to explore the equivalence of fractions.

Sources

Herani Tri Lestiana. *Promoting students’ understanding of the addition of fractions.* Master thesis, Universitas Negeri Surabaya, 2014.
www. fisme.science.uu.nl/en/impome/

Exploring the relation between fractions and percentages

Yenny Anggreini Sarumaha

The activity

The teacher tells about a match between two football clubs, Persebaya Surabaya and Sriwijaya FC. The fans of Persebaya Surabaya made a lot of noise during the game. After the game, the coach of the club says: 'We had a greater percentage of fans in the stands this time compared to the last game.' The team captain does not agree, however. He says: 'No, I think they just made more noise this time because we were winning.' The teacher discusses with the students what the coach and the team captain could have meant. Then she gives the numbers for that match:

> On the previous game, there were 1,350 spectators, of which 450 were Persebaya fans.
> This day, there were 1,600 spectators, and 500 of these were Persebaya fans.

The students work in small groups to figure out who was right, the coach or the team captain. They prepare a poster on which they write their calculations and their arguments.

Materials

- Paper for making calculations and larger paper for presenting the results.

For which students is this activity?

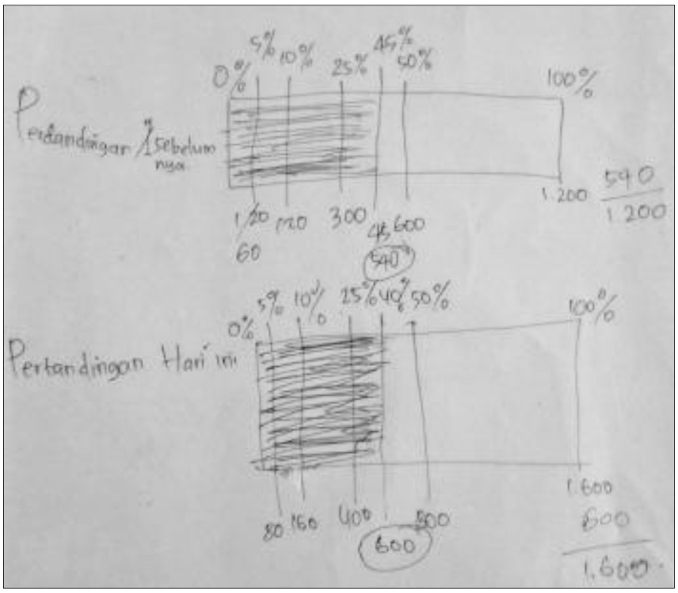
The activity is designed for students in grade 5. They should be familiar already with fractions and percentages. In this activity it is left to them which mathematical tool they choose to solve the problem.

Why is this an interesting activity?

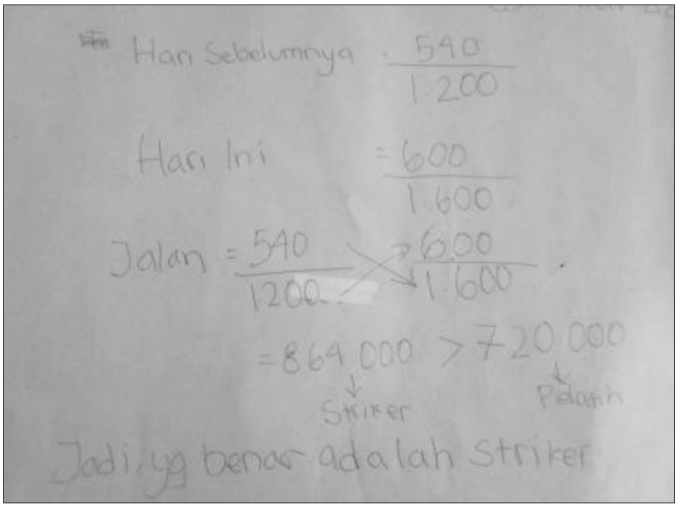
This is an open problem that can be solved in many different ways: by using percentages, by using fractions or by comparing ratio's. The variety of answers can be used as a starting point for a discussion of the relations between percentages, fractions and ratio's. It also offers an opportunity to discuss the calculation procedures.

What to expect and how to support the students?

The first question will be how students interpret the situation. In the design study some students argued that only the absolute number of fans did matter: last time there were 450 fans and this time 500. The coach of Persebaya Surabaya, however, seemed to hold a different view: he compared the noise made by the fans to the noise of the total crowd and decided that there must have been relatively more fans in the crowd. Was that the case indeed? In the design study the students had learned to draw a percentage bar as a way to work with percentages. An example can be seen below. The number of spectators is drawn as a bar with the total number written below and 100% written above the end point of the bar. Working stepwise the students found other percentages: if 1600 is 100%, then 10% is 160 people and 5% is 80 people. Reasoning this way, the group of students found that 540 out of 1200 people equals 45%. For the other match they also used the percentage bar, but they did not find the right answer. They were able to explain their calculations, however.



Other students interpreted the numbers in terms of fractions: which fraction is bigger, 540/1200 or 600/1600? The picture on the next page gives an example. In this case, however, one might question in how far the students understood the procedure they used; for the students cross multiplication is often a trick they have learned, but do not really understand. In any case the problem will lead to the question why the situation can be interpreted both in terms of percentages and in terms of fractions. The answer to that is that percentages are indeed a special kind of fractions; they are fractions with 100 as the denominator. A discussion about these relations is important, as the procedures students learn to work with fractions are different from those for working with percentages,



and therefore students may see fractions and percentages as something completely different.

A longer series of lessons

This lesson is the fifth in a series of lessons in which students explore percentages. Students often use procedures for working with fractions, percentages and proportions that they not really understand. In this series of lessons, therefore, all problems are given within context situations and the students learn to calculate percentages with a tool that assures that calculations remain connected to these contexts.

Sources

Yenny Anggreini Sarumaha. *Design research on mathematics education: investigating the development of Indonesian fifth grade students in learning percentages.* Master thesis Universitas Sriwijaya, 2012. See: www. fisme.science.uu.nl/en/impome/

van Galen, F.H.J. & van Eerde, H.A.A. (2013). Solving Problems with The Percentage Bar. *IndoMS Journal on Mathematics Education*, 4 (1), (pp. 1-8) (8 p.).

Developing students' understanding of the median and mean

Said Fachry Assagaf

The activity

Each student makes a hoop glider (see worksheet). In groups of 3 or 4 the students test which of their gliders flies best. Then the same person of the group throws the best glider 5 times. The students measure the distance it has travelled and write their data on the worksheet. The core problem in this lesson is to use the data of the five throws to predict how far the glider would fly the next time.

Materials

- Drinking straws, stiff paper and tape for each student to make a hoop glider.
- For each group a piece of rope with a mark at every meter (a knot, a piece of tape, or a mark made with a marker pen). This can be used in combination with a ruler to measure the flying distance of the glider.
- The worksheet.

For which students is this activity?

The lesson series was tested with students in grade 5. According to the Indonesian curriculum the topic of median and mean should be taught in grade 6.

Why is this an interesting activity?

The arithmetic mean (or shortly 'mean') is one of the statistical measures to describe a set of data. Other measures of central tendency are the median and the mode. Many students know the algorithm for computing the mean, without understanding the concept behind this formula. The activity that is described here raises the question how a set of data can be used to make predictions. In the discussions some groups may choose the median, wheras others may choose the arithmetic mean and both are defendable in this situation. In the lesson series a follow-up activity focuses on the arithmetic mean.

What to expect and how to support the students?

After the students have tested their glider, some agreement is necessary about how to do the measurements. The same person should throw the glider, using more or less the same force each time. The distance should be measured from the point where the glider was thrown till the point where it hits the ground. If the floor is smooth, this point may not be so easy to establish. In that case one may place some observers at the side of the room. It is easier if the planes will land in a sand box or on a carpet.

After the groups have gathered their data, let them discuss how they can use their data to predict how far the glider will fly a next time. Let a few of the groups present their data and explain their prediction. If a group has chosen, for example, the middle number (median), ask which group has reasoned differently. Be prepared, however, that many different arguments may be used. The group may discard, for example, trials where the plane did fly badly, or they may see a trend in their data (the distance becomes longer with every throw).

When it is clear how the groups have reasoned, ask how they would act if they had the data of, say, 100 throws instead of only 5. With only 5 throws luck plays an important role, but that would be less so with many more throws. Also let the students discuss if their strategy would be useful for comparing the gliders of the different groups ('which group has made the best one'?).

The activity will probably lead to all kinds of discussions and it is not necessary to reach a final decission on which strategy is the best. The lesson is a success if the students understand that in certain situations a measure for 'average' (not necessarily the mean) will be useful.

A longer series of lessons

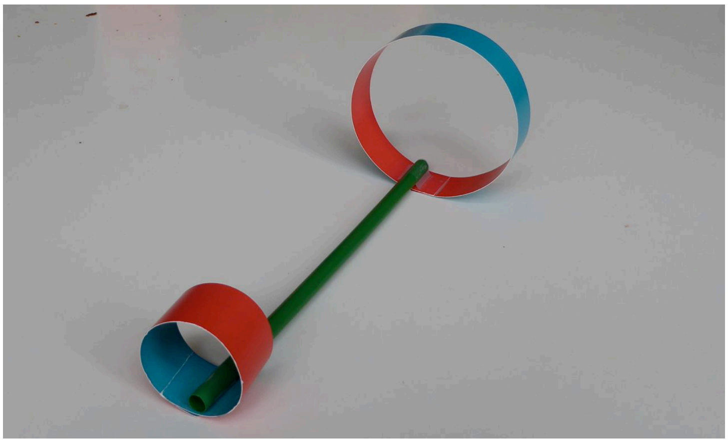
In the lesson series the next lesson uses the same context of flying gliders to focus on the procedure for finding the arithmetic mean. The students represent distances with bars and they use a compensation strategy to choose a number 'in the middle': find the difference between two bars, cut off half of the difference from the longest bar and add this to the shortest. The lesson series uses also other contexts: the weight of apples, the height of people and repeated measurement of the weight of an object.

Sources

Said Fachry Assagaf, *Developing the 5th grade students' understanding of the concept of mean through measuring activities.* Universitas Negeri Surabaya.
www. fisme.science.uu.nl/en/impome/

How well does your hoop glider fly?

1. To make a hoop glider you need thick paper or carton, a drinking straw and tape.
   Make a long strip of 2,5 cm x 30 cm and a short one of 2,5 cm x 15 cm.
   Use tape to make two hoops and attach these to the straw as shown in the picture.
   Throw the glider with the small hoop in front.



2. Work in small groups. Test each glider you have made and pick the one that seems to fly best.

3. Let one of you throw the glider 5 times, keeping the situation identical as much as possible: the same person throws the glider 5 times, using the same force. Measure the distance it has flown and write the data below.

1st throw	cm
2nd throw	cm
3rd throw	cm
4th throw	cm
5th throw	cm

4. Discuss with your group members the following.
   You have data of five actual throws; what is a good prediction for the distance that the glider will flow if you would throw it another time? Of course you cannot predict that distance precisely, it will be an estimation.
   Write down the number you agree on and write down your arguments for choosing that number.

Supporting the development of splitting in multiplication

Meryansumayeka

The activity

This activity is meant to stimulate students to use multiplication facts they already know when they have to find the answer to a multiplication with bigger numbers.
Within a story about Kondangan, groups of students are given 7 boxes that are supposed to contain 6 spoons each. The students are asked to arrange all boxes with spoons into two piles and to draw their arrangement. Then they are asked to find the total amount of spoons.
The questions on the worksheet are meant to be discussed in the small groups first. Later they should be discussed with the class as a whole.

Materials

- The worksheet
- (Empty) spoon boxes with the text '6 spoons'

For which students is this activity?

In the design study a series of activites was developed for students in grade 3. This introductory activity, however, can also be done with students in grade 2.

Why is this an interesting activity?

The activity makes students aware of the possibility to use a splitting strategy when they have to find the outcome of a multiplication.

Students thinking and teachers support

Students may come up with different arrangements of the 7 boxes. They might try at first to arrange the boxes in two equal piles, but this is impossible, as the number of boxes is odd. A likely arrangement will be to split the boxes into 4 and 3 boxes, so the piles are almost the same. The teacher then asks what other arrangements are possible. The students are asked to draw different arrangements.
Then the students have to calculate the total amount of spoons. Some students will directly calculate the total amount (7x6), others will calculate the amount in the two piles (for example 4x6 and 3x6) and then add the numbers.
The discussion should lead to the conclusion that the different ways of splitting 7x6 will result in different additions, but with the same outcome. This suggests a way to find the outcome of multiplications with bigger numbers: split the multiplication into multiplications with smaller numbers.

A longer series of lessons

This lesson is the first lesson in a longer series that aims at suporting students to use splitting strategies to solve multiplications.

Sources

Meryansumayeka. *Design research on multiplication: structures supporting the development of splitting level at grade 3 in indonesian primary school.* Master thesis Universitas Sriwijaya, 2011
www. fisme.science.uu.nl/en/impome/

1. After Kondangan was held, mother kept all the spoons that were used in boxes.
   There were 7 boxes, and in each box there were 6 spoons. Arrange those boxes into two pile of boxes.
   Draw your arrangement of spoon boxes.

2. Look at your drawing.
   How many spoons are in the first pile? Which multiplication can you use?

   How many spoons are in the second pile? Which multiplication can you use?

   How many spoons are there altogether? How can you find that answer?

3. Now draw another arrangement of the 7 boxes.

4. Answer the same questions as before:
   How many spoons are in the first pile? Which multiplication can you use?

   How many spoons are in the second pile? Which multiplication can you use?

   How many spoons are there altogether? How can you find that answer?